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Implicit function based adaptive control of non-canonical form discrete-time nonlinear systems☆

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ABSTRACT

This paper presents a new study on adaptive state feedback output tracking control problem for uncertain discrete-time nonlinear systems in a general non-canonical form. Time-advance operations on the output of such systems result in the output dynamics being nonlinearly dependent on the control input and unknown parameters, which leads to three technical issues: implicit relative degree; nonlinearly parameterized uncertainties; and non-affine control input. To address these issues, this paper first employs feedback linearization and implicit function theory to construct a relative degree dependent normal form; then proposes an adaptive parametric reconstruction based method to simultaneously deal with linearly and nonlinearly parameterized uncertainties in the output dynamics; and finally constructs a key implicit function equation to derive a unique adaptive control law which ensures closed-loop stability and asymptotic output tracking. An explicitly iterative solution based adaptive control law is also proposed to ensure closed-loop stability and bounded output tracking within any degree of accuracy. The simulation verifies the effectiveness of the proposed adaptive control method.

1. Introduction

Adaptive control is a powerful technique to deal with unknown parameters in control systems. Adaptive control design and analysis have been extensively studied. For example, Astolfi et al. (2008), Ioannou and Sun (2012), Krstic et al. (1995), Landau et al. (2011), Lavretsky and Wise (2013), Narendra and Annaswamy (1989), Sastry and Bodson (1989), Spooner et al. (2004), Tao (2003) and Zhou and Wen (2008), are some monographs addressing adaptive control of continuous-time (CT) and discrete-time (DT) control systems. Particularly, Sastry and Bodson (1989) is the first monograph on adaptive control of non-canonical form CT nonlinear systems with linearly parameterized uncertainties by using the feedback linearization technique; and Krstic et al. (1995) is the first monograph which systematically addressed adaptive control of strict-feedback form CT nonlinear systems. Adaptive control is also well developed for stochastic systems, for instance, Chen and Guo (2012), Goodwin and Sin (1984), Jiang and Xie (2021), Li, Deng, and Zhao (2019), Liu et al. (2007) and Zhao et al. (2015). There are too many remarkable results to list them all.

Most of the existing adaptive control methods are focused on canonical-form nonlinear systems which have explicit relative degrees and infinite zero structures, for instance, Bechlioulis and Rovithakis (2010), Chen et al. (2015), Fu et al. (2020), Krstic and Bement (2006), Lei et al. (2019), Li, Zhao, He, and Lu (2019), Lin and Qian (2001), Liu and Tong (2017), Niu et al. (2018), Sun et al. (2017), Wang et al. (2017, 2020, 2018), Xu et al. (2017), Yang et al. (2017) and Yu et al. (2021). However, many practical applications, such as aircraft flight control systems, their system dynamics structures are generally of non-canonical forms (Tao, 2014). Thus, it is of great importance to develop adaptive control algorithms to effectively and adaptively control such systems. Up to now, there have been several results to address adaptive output tracking problems for non-canonical

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form nonlinear systems. For instance, Sastry and Isidori (1989) established an adaptive output tracking control framework for non-canonical form CT nonlinear systems with linearly parameterized uncertainties. In recent years, adaptive control problems for non-canonical form CT nonlinear systems with unparameterizable uncertainties were also studied, using function approximation techniques (see, e.g., Zhang et al., 2019). By contrast, adaptive control of non-canonical form DT nonlinear systems with linearly parameterized uncertainties is seldom addressed, although feedback linearization of non-canonical DT nonlinear systems has been early addressed in Monaco and Normand-Cyrot (1987).

Note that the adaptive control methods for CT systems cannot be extended to DT systems due to essentially different properties induced by feedback linearization. As shown in Sastry and Isidori (1989), feedback linearization of non-canonical form CT nonlinear systems can define explicit relative degrees and the derived output dynamics linearly depends on the input and can be linearly parameterized. However, as will be shown in Section 2, feedback linearization of non-canonical form DT nonlinear systems can only define implicit relative degrees and the derived output dynamics has new features: nonlinear dependence on system parameters and the input. In other words, the adaptive control problem faces new technical issues: implicit relative degree; nonlinearly parameterized uncertainties; and non-affine control input. As early as 1989, in Sastry and Isidori (1989), the authors have pointed out these issues. However, up to now, there are still no valid results to solve the adaptive control of non-canonical form DT nonlinear systems.

In this paper, we will develop an implicit function based solution to the adaptive control problem of general non-canonical DT nonlinear systems. The main contributions are summarized as follows.

- It establishes an implicit function based adaptive control framework which solves the new technical issues in adaptive control of non-canonical form DT nonlinear systems: implicit relative degrees, non-affine control input, and nonlinearly parameterized uncertainties, and achieves desired system performance. The above three issues have not ever been simultaneously addressed in the literature.
- It proposes an adaptive parametric reconstruction based method which effectively deals with all unknown parameters including linearly and nonlinearly parameterized uncertainties in the output dynamics.
- It derives a unique adaptive control law from an implicit function equation, which ensures closed-loop stability and asymptotic output tracking for the controlled plant. The uniqueness of the adaptive control law is addressed in this paper, which is seldom discussed in the literature of adaptive control.
- It constructs an explicitly iterative solution based adaptive control law which is easy to be implemented in practice and can be used for the case when the analytical adaptive control law is difficult to obtain.

The rest of this paper is organized as follows. Section 2 gives the controlled plant and technical issues. Section 3 discusses the relative degree issue for the controlled plant. Section 4 presents a nominal control framework to show some fundamentals. Section 5 shows the adaptive control design details. Sections 6 and 7 give the simulation study and concluding remarks, respectively.

2. Problem statement

In this section, we give the system model and clarify the technical issues to be solved.

2.1. System model

Consider the following DT nonlinear system:

$$x(t+1)=f(x(t))+g(x(t))u(t),\ \ y(t)=Cx(t), \tag{1}$$

where $t\in\{0,1,2,\ldots\}$; $x(t)=[x_1(t),x_2(t)\ldots,x_n(t)]^T\in\mathbb{R}^n$ is the state vector, $u(t)\in\mathbb{R}$ is the control input, and $y(t)\in\mathbb{R}$ is the system output; $C^T=[c_1,c_2,\ldots,c_n]^T\in\mathbb{R}^n$ is an unknown constant parameter vector; and $f:\mathbb{R}^n\to\mathbb{R}^n$ and $g:\mathbb{R}^n\to\mathbb{R}^n$ are sufficiently smooth and nonlinear mappings of the forms $f(x(t))=[f_1(x(t)),f_2(x(t)),\ldots,f_n(x(t))]^T$, $g(x(t))=[g_1(x(t)),g_2(x(t)),\ldots,g_n(x(t))]^T$. Moreover,

$$f_i(x(t))=\sum_{j=1}^{p_i}\theta_{ij}^{1*}f_{ij}(x(t)),\,g_i(x(t))=\sum_{j=1}^{q_i}\theta_{ij}^{2*}g_{ij}(x(t)) \tag{2}$$

for some known positive integers p_i and q_i, where $\theta_{ij}^{1*}\in\mathbb{R}$ and $\theta_{ij}^{2*}\in\mathbb{R}$ are unknown constant parameters; and $f_{ij}:\mathbb{R}^n\to\mathbb{R}$ and $g_{ij}:\mathbb{R}^n\to\mathbb{R}$ are known and nonlinear mappings with $f_{ij}(0)=0$. In this paper, the system state is assumed to be measurable.

2.2. Technical issues to be solved

The control objective is to develop an adaptive state feedback control scheme to ensure closed-loop stability and asymptotic output tracking for the non-canonical form DT nonlinear system (1).

Since (1) is in a non-canonical form, it is not suitable for adaptive control design, and needs to be reconstructed in prior. The powerful feedback linearization technique is a natural choice to reconstruct the non-canonical system dynamics. Now, we show that, after a feedback linearization based reconstruction, the system dynamics will involve the issues of nonlinear parametrization and the control input in a non-affine form.

According to the relative degrees of DT nonlinear systems defined in Monaco and Normand-Cyrot (1987), if the system (1) has relative degree 1 on $\mathbb{R}^n$, then

$$y(t+1)=\sum_{i=1}^{n}\sum_{j=1}^{p_i}c_i\theta_{ij}^{1*}f_{ij}(x(t))+\sum_{i=1}^{n}\sum_{j=1}^{q_i}c_i\theta_{ij}^{2*}g_{ij}(x(t))u(t) \tag{3}$$

with $\sum_{i=1}^{n}\sum_{j=1}^{q_i}c_i\theta_{ij}^{2*}g_{ij}(x)\neq 0,\forall x\in\mathbb{R}^n$. If the system (1) has relative degree 2 on $\mathbb{R}^n$, then $y(t+1)=\sum_{i=1}^{n}\sum_{j=1}^{p_i}c_i\theta_{ij}^{1*}f_{ij}(x(t))$ and

$$y(t+2)=\sum_{i=1}^{n}\sum_{j=1}^{p_i}c_i\theta_{ij}^{1*}f_{ij}(x(t+1)) \tag{4}$$

for $x(t+1)=[x_1(t+1),x_2(t+1),\ldots,x_n(t+1)]^T$ with

$$x_i(t+1)=\sum_{j=1}^{p_i}\theta_{ij}^{1*}f_{ij}(x(t))+\sum_{j=1}^{q_i}\theta_{ij}^{2*}g_{ij}(x(t))u(t) \tag{5}$$

such that $\sum_{i=1}^{n}\sum_{j=1}^{p_i}\sum_{k=1}^{n}\sum_{l=1}^{q_k}c_i\theta_{ij}^{1*}\theta_{kl}^{2*}\frac{\partial f_{ij}(x(t+1))}{\partial x_k(t+1)}g_{kl}\neq 0$. Note that f_{ij} are nonlinear mappings. In addition to (4) and (5), θ_{ij}^{1*}, θ_{ij}^{2*} and $u(t)$ nonlinearly exist in $y(t+2)$. Such a characteristic brings two difficulties for adaptive control design: how to handle the unknown parameters in $y(t+2)$; and how to derive an effective adaptive control law to achieve desired system performance.

Canonical-form DT nonlinear systems are generally of the basic form $x_i(t+1)=f_i(\bar{x}_i)+g_i(\bar{x}_i)x_{i+1}$, $x_n(t+1)=f_n(\bar{x}_n)+g_n(\bar{x}_n)u$, $y=x_1$, where $i=1,2,\ldots,n-1$, $\bar{x}_j=[x_1,x_2,\ldots,x_j]^T\in\mathbb{R}^j$, which have explicitly certain relative degrees, and the adaptive control problem can be solved based on the adaptive backstepping technique (Krstic et al., 1995). However, non-canonical form

DT nonlinear systems do not fulfill the canonical-form matching condition. On the other hand, as clarified in Introduction, the control methods for CT systems are also not applicable to the new control problem addressed in this paper.

In summary, we conclude that adaptive control of non-canonical form DT nonlinear systems is still open for study. To meet the control objective, we need to solve the following technical issues:

- how to handle linearly and nonlinearly parameterized uncertainties in the output dynamics of the system (1) with high-order relative degree;
- how to develop an analytical adaptive state feedback control law for the system (1) to achieve closed-loop stability and asymptotic output tracking; and
- how to develop an iterative solution based adaptive control law for the system (1) to achieve closed-loop stability and bounded output tracking with an arbitrary degree of accuracy for the case when the analytical adaptive control law is difficult to obtain.

3. Relative degrees and normal form

This section discusses relative degrees and normal form of the system (1). A design condition for tracking control is made on the normal form.

3.1. Relative degrees of DT nonlinear systems

Introduce $\circ$ to denote a composition operation, that is, $p_1 \circ p_2$ denotes that p_1 is a function of p_2 for any functions p_1 and p_2 of appropriate dimensions. Define

$$F_i(x(t), u(t)) = \sum_{j=1}^{p_i} \theta_{ij}^{1*} f_{ij}(x(t)) + \sum_{j=1}^{q_i} \theta_{ij}^{2*} g_{ij}(x(t))u(t),$$

which implies $F_i(x, 0) = \sum_{j=1}^{p_i} \theta_{ij}^{1*} f_{ij}(x)$. Let $F(x, u) = [F_1(x, u), \ldots, F_n(x, u)]^T$, $F_0(x) = [F_1(x, 0), \ldots, F_n(x, 0)]^T$. Then, for any integer $k > 0$, $F_0^k \circ F(x, u) = F_0(F_0^{k-1} \circ F(x, u))$ and $F_0^0 \circ F(x, u) = F(x, u)$. Now, we specify a general relative degree for the system (1) as follows.

Lemma 1. *The system (1) has relative degree ρ $(1 \le \rho \le n)$ on $\mathbb{R}^n \times \mathbb{R}$, if, for $k = 0, 1, \ldots, \rho - 2$, $\forall (x, u) \in \mathbb{R}^n \times \mathbb{R}$,*

$$C \frac{\partial F_0^k \circ F(x, u)}{\partial u} = 0, \ \ C \frac{\partial F_0^{\rho-1} \circ F(x, u)}{\partial u} \neq 0.$$

The proof of this lemma can be obtained based on the relative degree definition of DT nonlinear systems developed in Monaco and Normand-Cyrot (1987), so we omit it. Next, based on the relative degree information, we construct a normal form for the system (1).

Remark 2. The relative degree of DT systems has the following physical explanation. In real control systems, the time delay generally occurs between the input and output. The relative degree of DT systems is actually the input–output time delay. Eq. (8) describes the exact time delay between $u(t)$ and $y(t)$. Specifically, when a non-zero input is applied to the system at the time instant t, the output has no response at the instants $t+1, t+2, \ldots, t+\rho-1$, but has response at the instant $t+\rho$. The relative degree (time delay) information is crucial for adaptive control design (Goodwin & Sin, 1984; Ioannou & Sun, 2012; Narendra & Annaswamy, 1989; Tao, 2003). □

3.2. Normal form of DT nonlinear systems

Now, we give the following result to specify a relative degree dependent normal form for the system (1).

Lemma 3. *If the system (1) has relative degree ρ for all $(x, u) \in \mathbb{R}^n \times \mathbb{R}$, via a diffeomorphism $T(x(t)) = [\xi^T(t), \eta^T(t)]^T$ with $\xi(t) = [\xi_1(t), \xi_2(t), \ldots, \xi_\rho(t)]^T \in \mathbb{R}^\rho$ and $\eta(t) \in \mathbb{R}^{n-\rho}$, the system (1) can be transformed into two subsystems: the output dynamics*

$$\xi_i(t+1) = \xi_{i+1}(t), i = 1, \ldots, \rho - 1,$$
$$\xi_\rho(t+1) = CF_0^{\rho-1} \circ F(x(t), u(t)) \quad (6)$$

with $\xi_1(t) = y(t)$ such that $C \frac{\partial F_0^{\rho-1} \circ F(x,u)}{\partial u} \neq 0$, $\forall (x, u) \in \mathbb{R}^n \times \mathbb{R}$, and the internal dynamics

$$\eta(t+1) = q(\xi(t), \eta(t), u(t)), \quad (7)$$

where $q : \mathbb{R}^\rho \times \mathbb{R}^{n-\rho} \times \mathbb{R} \to \mathbb{R}^{n-\rho}$ is a smooth mapping.

The proof of this lemma is similar to that of CT system case, and the readers may refer to Isidori (1995) for study. We do not provide the details for simplicity. Based on Lemma 3, we have

$$y(t+\rho) = CF_0^{\rho-1} \circ F(x(t), u(t)) \quad (8)$$

which will be used for adaptive control design. For stable output tracking control, we need a design condition on the internal dynamics (7), which is specified as follows.

3.3. Input-to-state stable (ISS) condition

For state feedback output tracking control design, the input is generically designed as $u(t) = u(x(t), v(t)) = u(T^{-1}(\xi(t), \eta(t)), v(t))$, where $v \in \mathbb{R}$ is bounded and independent of x. Therefore, (7) is rewritten as

$$\eta(t+1) = Q(\xi(t), \eta(t), v(t)), \quad (9)$$

where $Q : \mathbb{R}^\rho \times \mathbb{R}^{n-\rho} \times \mathbb{R} \to \mathbb{R}^{n-\rho}$ is a smooth mapping. Then, we make the following assumption.

Assumption 1. The dynamic system $\eta(t+1) = Q(0, \eta(t), 0)$ is exponentially stable, and $Q(\xi(t), \eta(t), v(t))$ is globally Lipschitz with respect to $\xi(t)$ and $v(t)$.

Under Assumption 1, from (9), one can verify that, if ξ and v are bounded, η is bounded. For adaptive output tracking control of nonlinear systems covering CT and DT, the ISS design condition is essential (for instance, Chen and Khalil 1995 and Ge and Zhang 2003). There are many applications whose internal dynamics are ISS, such as the NASA Generic Transport Model shown in Guo et al. (2011). The ISS design condition can be seen as an extension of the fundamental condition that requires the zeros of the transfer function are stable for adaptive control of linear time-invariant systems.

4. Nominal control framework

Before proceeding adaptive control design, we present a nominal control framework, assuming all system parameters were known, to show some fundamentals.

Note that the relative degree condition in Lemma 1 ensures $C \frac{\partial F_0^{\rho-1} \circ F(x,u)}{\partial u} \neq 0$ for any finite x and u. Adaptive control design further requires that $C \frac{\partial F_0^{\rho-1} \circ F(x,u)}{\partial u}$ is non-zero when x and u go to infinity. Thus, we make the following assumption.

Assumption 2. There exists some constant ε such that $|C\frac{\partial F_0^{\rho-1}\circ F(x,u)}{\partial u}| \geq \varepsilon > 0$, $\forall (x,u) \in \mathbb{R}^n \times \mathbb{R}$, and the sign of $C\frac{\partial F_0^{\rho-1}\circ F(x,u)}{\partial u}$ is known.

Remark 4. The condition $|C\frac{\partial F_0^{\rho-1}\circ F(x,u)}{\partial u}| \geq \varepsilon$ can be seen as a modified relative degree condition. The sign of $C\frac{\partial F_0^{\rho-1}\circ F(x,u)}{\partial u}$ to be known means the control direction is known. Without loss of generality, we assume that the sign is positive. Nussbaum and multiple-model techniques are often used to relax the control gain sign condition (Chen et al., 2019; Ge & Wang, 2003). □

Output dynamics. For the system (1) with relative degree ρ on $\mathbb{R}^n \times \mathbb{R}$, it follows from (6) and (8) that the output dynamics can be expressed as

$$y(t+\rho) = CF_0^{\rho-1} \circ F(x(t), u(t)). \tag{10}$$

Implicit function. Introduce an implicit function as

$$\varphi(x(t), u(t), y_m(t+\rho)) = CF_0^{\rho-1} \circ F(x(t), u(t)) - y_m(t+\rho), \tag{11}$$

where $y_m(t+\rho)$ is ρ-step time advance of a given reference output. Based on the implicit function definition (Krantz & Parks, 2002), to ensure that $\varphi(x,u,y_m)$ in (11) is a well-defined implicit function of u, the signal y_m needs to satisfy that y_m belongs to the range of $CF_0^{\rho-1} \circ F(x,u)$ for all $t = 0, 1, 2, \ldots$.

Motivated by the implicit function result (Zhang & Ge, 2006), we derive the following result which specifies an implicit function based nominal control framework.

Theorem 5. *Under Assumptions 1–2, if the system (1) has relative degree ρ for all $(x,u) \in \mathbb{R}^n \times \mathbb{R}$, there exists a unique state feedback control law that ensures closed-loop stability and output tracking $y(t+\rho) = y_m(t+\rho)$.*

Proof. We first show that there exists a unique solution u to the following equation

$$CF_0^{\rho-1} \circ F(x(t), u) - y_m(t+\rho) = 0. \tag{12}$$

Under Assumption 2, fixing $x(t)$ and $y_m(t+\rho)$, for all $u \in \mathbb{R}$, there exists a bounded signal denoted as $\underline{\kappa}(t)$ such that $\underline{\kappa}(t) = \inf_{u\in\mathbb{R}}\left\{\frac{\partial\varphi(x(t),u,y_m(t+\rho))}{\partial u}\right\} > 0$. Let

$$c(t) = \varphi(x(t), u, y_m(t+\rho))|_{u=0}. \tag{13}$$

If $c(t) = 0$, then $u = 0$ is the solution. Otherwise, consider a compact set defined as $\Omega_t = \{u||u| \leq |c(t)|/\underline{\kappa}(t)\}$. Now, due to the continuity of $\frac{\partial\varphi(x(t),u,y_m(t+\rho))}{\partial u}$, fixing $x(t)$ and $y_m(t+\rho)$, we conclude that $\frac{\partial\varphi(x(t),u,y_m(t+\rho))}{\partial u}$ has a maximum value for all $u \in \Omega_t$. In other words, fixing $x(t)$ and $y_m(t+\rho)$, there exists a bounded signal depending on t, denoted as $\bar{\kappa}(t)$, such that $\bar{\kappa}(t) = \max_{u\in\Omega_t}\left\{\frac{\partial\varphi(x(t),u,y_m(t+\rho))}{\partial u}\right\} > 0$. Introduce a mapping $f_t : \Omega_t \to \mathbb{R}$ defined as

$$f_t(u) = u - \frac{1}{\gamma(t)}\varphi(x(t), u, y_m(t+\rho)),$$

where $\gamma(t)$ is a design parameter depending on t such that $\gamma(t) > \bar{\kappa}(t)$. For any $u \in \Omega_t$, it follows from Mean Value Theorem that

$$\varphi(x(t), u, y_m(t+\rho)) = c(t) + u\frac{\partial\varphi(x(t), u, y_m(t+\rho))}{\partial u}|_{u=\sigma_1}, \tag{14}$$

where σ_1 is a value depending on t such that $|\sigma_1| < |u|$. Thus, (13)–(14) yield

$$\begin{aligned}
&|f_t(u)| \\
&= \left|u - \frac{1}{\gamma(t)}\left(c(t) + u\frac{\partial\varphi(x(t), u, y_m(t+\rho))}{\partial u}|_{u=\sigma_1}\right)\right| \\
&\leq \left|1 - \frac{1}{\gamma(t)}\frac{\partial\varphi(x(t), u, y_m(t+\rho))}{\partial u}|_{u=\sigma_1}\right||u| + \frac{|c(t)|}{\gamma(t)}.
\end{aligned}$$

Since $u, \sigma_1 \in \Omega_t$, with $\gamma(t) > \bar{\kappa}(t) \geq \underline{\kappa}(t)$, we derive that $|f_t(u)| \leq \frac{|c(t)|}{\gamma(t)} + \left(1 - \frac{\underline{\kappa}(t)}{\gamma(t)}\right)\frac{|c(t)|}{\underline{\kappa}(t)} = \frac{|c(t)|}{\underline{\kappa}(t)}$ which implies that f_t maps Ω_t into itself. In addition, for any $u_1, u_2 \in \Omega_t$, we have

$$|f_t(u_1) - f_t(u_2)| = |u_1 - u_2 - \frac{1}{\gamma(t)}(\varphi|_{u=u_1} - \varphi|_{u=u_2})|$$

which follows from Mean Value Theorem that

$$\begin{aligned}
|f_t(u_1) - f_t(u_2)| &\leq \left|1 - \frac{1}{\gamma(t)}\frac{\partial\varphi}{\partial u}|_{u=\sigma_2}\right||u_1 - u_2| \\
&\leq \left(1 - \frac{\underline{\kappa}(t)}{\gamma(t)}\right)|u_1 - u_2|,
\end{aligned} \tag{15}$$

where σ_2 is some value between u_1 and u_2. From (15), we see that f_t is a contraction mapping. Then, it follows from Banach's Fixed Point Theorem that there exists a unique solution denoted as u_t^* to the equation $f_t(u) = u$, that is, $u_t^* - \frac{1}{\gamma(t)}\varphi(x(t), u_t^*, y_m(t+\rho)) = u_t^*$, which implies that u_t^* is the unique solution to $\varphi(x(t), u, y_m(t+\rho)) = 0$. Then, from (11), we derive that u_t^* is the unique solution to Eq. (12). If u_t^* is set as the control law, then Eq. (12) always holds. In addition to (10), we obtain $y(t+\rho) - y_m(t+\rho) = 0$.

Since $y_m(t) \in L^\infty$, we have $\xi(t) = [y(t), y(t+1), \ldots, y(t+\rho-1)]^T \in L^\infty$. Under Assumption 1, the boundedness of $\xi(t)$ implies that of $\eta(t)$, which follows from the diffeomorphism $T(x) = [\xi^T, \eta^T]^T$ that $x \in L^\infty$. As u_t^* belongs to Ω_t, we obtain the control law is also bounded. Thus, all closed-loop signals are bounded. ∇

Theorem 5 provides a basic nominal control scheme. To derive the nominal control law, a feasible way, in practice, is to solve (12) so as to get a solution $u(t)$. As $u(t)$ nonlinearly exists in (12), it may be difficult to obtain. Alternatively, we can design a sequence $\{u_i(t)\}$: $u_i(t) = u_{i-1}(t) - \frac{1}{\gamma(t)}\varphi(x(t), u(t-1), y_m(t+\rho))$, $i = 1, 2, \ldots$, with $u_0(t) = u(t-1)$, where $u(t-1)$ denotes the control law at the instant $t-1$. Following a similar analysis in the proof of Theorem 5, one can verify that $\{u_i(t)\}$ is convergent to u_t^* for all $t \geq 1$. Thus, one can use an iterative solution based control law to ensure bounded output tracking. We do not provide the details for space.

5. Adaptive control designs

In this section, we first present a brief adaptive control framework for the system (1) with $\rho = 1$. Then, we extensively address the adaptive control design for the system (1) with $\rho = 2$. Finally, we illustrate the $\rho = 3$ case to demonstrate the control design for the $\rho = 2$ case can be extended to any higher-order relative degree case.

5.1. Adaptive control design for systems with $\rho = 1$

Adaptive control design for the system (1) with $\rho = 1$ has the following details.

Parameterized model. For the system (1) with $\rho = 1$, from (3), a parameterized model for (3) is derived as

$$y(t+1) = \theta^{1*T}\omega_1(t) + \theta^{2*T}\omega_2(t)u(t), \tag{16}$$

where $\theta^{1*} \in \mathbb{R}^{\sum_{i=1}^n p_i}$ and $\theta^{2*} \in \mathbb{R}^{\sum_{i=1}^n q_i}$ are unknown constant vectors, $\omega_1 \in \mathbb{R}^{\sum_{i=1}^n p_i}$ and $\omega_2 \in \mathbb{R}^{\sum_{i=1}^n q_i}$ are known time-varying regressor vectors, and

$$\theta^{1*} = [c_1\theta_{11}^{1\ *}, \dots, c_1\theta_{1p_1}^{1\ *}, \dots, c_n\theta_{n1}^{1\ *}, \dots, c_n\theta_{np_n}^{1\ *}]^T,$$
$$\theta^{2*} = [c_1\theta_{11}^{2\ *}, \dots, c_1\theta_{1q_1}^{2\ *}, \dots, c_n\theta_{n1}^{2\ *}, \dots, c_n\theta_{nq_n}^{2\ *}]^T,$$
$$\omega_1(t) = [f_{11}(x(t)), \dots, f_{1p_1}(x(t)), \dots, f_{n1}(x(t)), \dots, f_{np_n}(x(t))]^T,$$
$$\omega_2(t) = [g_{11}(x(t)), \dots, g_{1q_1}(x(t)), \dots, g_{n1}(x(t)), \dots, g_{nq_n}(x(t))]^T.$$

Adaptive control law. The adaptive law is designed as

$$u(t) = \frac{1}{\theta^{2^T}(t)\omega_2(t)}(-\theta^{1^T}(t)\omega_1(t) + y_m(t+1)), \tag{17}$$

where $\theta^1(t)$ and $\theta^2(t)$ are estimates of θ^{1*} and θ^{2*}, respectively. Note that $y_m(t)$ is a given reference output and $y_m(t+1)$ is available at the current time instant.

Tracking error equation. Define the tracking error as $e(t) = y(t) - y_m(t)$. Then, substituting the adaptive control law (17) to (16) yields $e(t+1) = -\tilde{\theta}^T(t)\omega(t)$, where $\tilde{\theta}(t) = \theta(t) - \theta^*$ with $\theta(t) = [\theta^{1^T}(t), \theta^{2^T}(t)]^T$ and $\theta^* = [\theta^{1*^T}, \theta^{2*^T}]^T$, and $\omega(t) = [\omega_1^T(t), \omega_2^T(t)]^T$.

Adaptive update law. Define an estimation error $\epsilon(t) = e(t) - \sigma(t)$ with $\sigma(t) = \theta^T(t)\omega(t-1) - \theta^T(t-1)\omega(t-1)$. The adaptive update law is designed as

$$\theta(t+1) = \theta(t) + \frac{\Gamma\epsilon(t)\omega(t-1)}{m^2(t)}, \ \theta(0) = \theta_0, \tag{18}$$

where $\Gamma = \mathrm{diag}\{\gamma_1, \dots, \gamma_{\sum_{i=1}^n(p_i+q_i)}\}$ is an adaptation gain matrix with $\gamma_i \in (0, 2)$, θ_0 is an initial estimate of θ^*, and $m(t) = \sqrt{1 + \omega^T(t-1)\omega(t-1) + \sigma^2(t)}$. To ensure a well-defined adaptive control law, $\theta^2(t)$ needs to be constrained to avoid $\theta^{2^T}(t)\omega_2(t) = 0$. A parameter projection is capable of handling this issue. One may refer to Cougnon et al. (2011), Tao (2003), and related literature. We do not provide the details.

Now, we give the following result:

Theorem 6. *Under Assumptions 1–2, the adaptive law (17) with the update law (18), applied to the system (1) with $\rho = 1$ and unknown $C, \theta_{ij}^{1*}, \theta_{ij}^{2*}$, $i = 1, 2, \dots, n;\ j = 1, 2, \dots, p_i(q_i)$, ensures closed-loop stability and asymptotic output tracking:* $\lim_{t\to\infty}(y(t) - y_m(t)) = 0$.

The proof of this theorem can be performed based on the stability analysis in Section 7.3 of the book (Tao, 2003). Here, we do not provide the details. This theorem reveals that adaptive control for the system with relative degree one can be solved by using a standard linear parametrization based formulation.

Extension to canonical-form DT systems with a general relative degree. In this part, we show that the adaptive control method for the $\rho = 1$ case is applicable to adaptive control of canonical-form DT nonlinear systems with a general relative degree.

Consider a canonical-form DT nonlinear system:

$$\xi_i(t+1) = \xi_{i+1}(t), i = 1, 2, \dots, \rho - 1, \tag{19}$$
$$\xi_\rho(t+1) = f(\xi(t), \eta(t)) + g(\xi(t), \eta(t))u(t), \tag{20}$$
$$\eta(t+1) = q(\xi(t), \eta(t), u(t)), \tag{21}$$
$$y(t) = \xi_1(t), \tag{22}$$

where the state vector is $[\xi^T(t), \eta^T(t)]^T \in \mathbb{R}^n$ with $\eta(t) \in \mathbb{R}^{n-\rho}$ and $\xi(t) = [\xi_1(t), \xi_2(t), \dots, \xi_\rho(t)]^T \in \mathbb{R}^\rho$, $u(t) \in \mathbb{R}$ is the input, $y(t) \in \mathbb{R}$ is the output; and $f \in \mathbb{R}, g \in \mathbb{R}, q \in \mathbb{R}^{n-\rho}$ are smooth functions with linearly parameterized uncertainties such that $g \geq \varepsilon$ for some constant $\varepsilon > 0$. We assume that the state is measurable; and an ISS condition, similar to Assumption 1, is made on the internal dynamics (21). Note that the system (19)–(22) is of a canonical-form with a general relative degree ρ.

Now, imitating the adaptive control design for the system (1) with $\rho = 1$, we give an outline of adaptive control design for the canonical-form system (19)–(22).

Step 1: Parametrization of output dynamics. Following the design procedure of the relative degree one case, a parameterized model of the output dynamics (19) should be first derived, which can be expressed as

$$y(t+\rho) = \theta_\rho^{1*^T}\omega_{\rho 1}(t) + \theta_\rho^{2*^T}\omega_{\rho 2}(t)u(t),$$

where θ_ρ^{1*} and θ_ρ^{2*} are unknown constant vectors and $\omega_{\rho 1}(t)$ and $\omega_{\rho 2}(t)$ are known time-varying regressor vectors all of appropriate dimensions.

Step 2: Specification of adaptive control law. Imitating (17), the adaptive control law for the system (19)–(21) has the following structure:

$$u(t) = \frac{1}{\theta_\rho^{2^T}(t)\omega_{\rho 2}(t)}(-\theta_\rho^{1^T}(t)\omega_{\rho 1}(t) + y_m(t+\rho)), \tag{23}$$

where $\theta_\rho^i(t)$, $i = 1, 2$, are estimates of θ^{i*}, and $y_m(t+\rho)$ is ρ-step time advance of a given reference output $y_m(t)$.

Step 3: Derivation of tracking error equation. Substituting (18) to (8) derives a tracking error equation as

$$e_\rho(t+1) = -\tilde{\theta}_\rho^T(t)\omega_\rho(t), \tag{24}$$

where $\tilde{\theta}_\rho(t) = \theta_\rho(t) - \theta_\rho^*$ with $\theta_\rho(t) = [\theta_\rho^{1^T}(t), \theta_\rho^{2^T}(t)]^T$ and $\theta_\rho^* = [\theta_\rho^{1*^T}, \theta_\rho^{2*^T}]^T$, and $\omega_\rho(t) = [\omega_{\rho 1}^T(t), \omega_{\rho 2}^T(t)]^T$.

With the adaptive control law (23) and the tracking error model (24), we can derive the parameter update law similar to (18), and finally prove that the adaptive control law (23) can ensure closed-loop stability and asymptotic output tracking for the system (19)–(22). The stability analysis is similar to that in Section 7.3 of the book Tao (2003). Here, we do not provide the details.

So far, we have shown that the adaptive control problem for the system (1) with $\rho = 1$ can be solved based on a standard adaptive control framework, and shown that the method is applicable to canonical-form DT nonlinear systems with a general relative degree. However, for the system (1) with $\rho > 1$, the adaptive control design is quite different from that of the $\rho = 1$ case, which will be shown subsequently.

5.2. Adaptive control design for systems with $\rho = 2$

As clarified in Sections 1 and 2, adaptive control for the $\rho > 1$ case involves nonlinear parametrization and non-affine control input, which cannot be solved by the existing control methods and needs systematic study. In this section, we will design an implicit function equation based adaptive control law to ensure stable output tracking under some additional conditions on the mappings f_{ij} and g_{ij} in (2).

For f_{ij} and g_{ij} in (2), we make the following assumption.

Assumption 3. (a) f_{ij} are globally Lipschitz; and (b) $g_{ij} \in L^\infty$; $i = 1, \dots, n, j = 1, \dots, p_i(q_i)$.

Assumption 3(a) is a common Lipschitz condition. Similar Lipschitz conditions are often used in the literature, e.g., Ge and Zhang (2003) and Sastry and Isidori (1989) for CT system case; and Beikzadeh and Marquez (2016) and Nguyen and Trinh (2016) for DT system case. Assumption 3(b) means the control coefficient in the system dynamics is bounded, which is reasonable for many applications, for instance, robot manipulator model (Lewis et al., 1993), rapid thermal processing (RTP) system model (Emami-Naeini et al., 2003), etc.

An outline. To deal with the unknown parameters in $y(t+2)$, we will propose an adaptive parametric reconstruction based method. The specific steps are (i) two auxiliary signals are introduced, which are available and can be linearly parameterized; (ii) the modified gradient algorithms are employed to derive some estimates of the unknown parameters in the parametrizations of auxiliary signals; and (iii) the derived estimates are the indirect estimates of the unknown parameters in $y(t+2)$. We then need to construct estimates of several future time signals, which, together with parameter estimates, are used to construct a key implicit function equation. Finally, the adaptive control law can be derived from solving the implicit function equation. The details are as follows.

Output dynamics. If the system (1) has relative degree two for $(x, u) \in \mathbb{R}^n \times \mathbb{R}$, the output dynamics is

$$y(t+2) = \sum_{i=1}^{n}\sum_{j=1}^{p_i} c_i\theta_{ij}^{1*} f_{ij}(x(t+1)) \tag{25}$$

such that $\sum_{i=1}^{n}\sum_{j=1}^{p_i}\sum_{k=1}^{n}\sum_{l=1}^{q_k} c_i\theta_{ij}^{1*}\theta_{kl}^{2*}\frac{\partial f_{ij}(x(t+1))}{\partial x_k(t+1)} g_{kl} \geq \varepsilon$. Define $\theta_y^* \in \mathbb{R}^{\sum_i^n p_i}$, $\Theta_x^* \in \mathbb{R}^{n\times\sum_{i=1}^n (p_i+q_i)}$ of the forms $\theta_y^* = [c_1\theta_{11}^{1*}, \ldots, c_1\theta_{1p_1}^{1*}, \ldots, c_n\theta_{n1}^{1*}, \ldots, c_n\theta_{np_n}^{1*}]^T$ and

$$\Theta_x^* = \begin{bmatrix} \theta_1^{1*T} & \theta_1^{2*T} & & & \\ & & \ddots & & \\ & & & \theta_n^{1*T} & \theta_n^{2*T} \end{bmatrix}, \tag{26}$$

with $\theta_i^{1*} = [\theta_{i1}^{1*}, \ldots, \theta_{ip_i}^{1*}]^T$ and $\theta_i^{2*} = [\theta_{i1}^{2*}, \ldots, \theta_{iq_i}^{2*}]^T$ such that all other elements (that are not given) of Θ_x^* are zero. Moreover, define

$$\bar{f}_i(x(t)) = [f_{i1}(x(t)), \ldots, f_{ip_i}(x(t))]^T \in \mathbb{R}^{p_i},$$
$$\bar{g}_i(x(t)) = [g_{i1}(x(t)), \ldots, g_{iq_i}(x(t))]^T \in \mathbb{R}^{q_i},$$
$$\phi_g(t) = [\bar{g}_1^T(x(t)), \ldots, \bar{g}_n^T(x(t))]^T \in \mathbb{R}^{\sum_{i=1}^n q_i},$$
$$\phi_f(t) = [\bar{f}_1^T(x(t)), \ldots, \bar{f}_n^T(x(t))]^T \in \mathbb{R}^{\sum_{i=1}^n p_i}, \tag{27}$$
$$\phi_{x_i}(t) = [\bar{f}_i^T(x(t)), \bar{g}_i^T(x(t))u(t)]^T \in \mathbb{R}^{p_i+q_i}, \tag{28}$$
$$\phi_x(t) = [\phi_{x_1}^T(t), \ldots, \phi_{x_n}^T(t)]^T \in \mathbb{R}^{\sum_{i=1}^n (p_i+q_i)}. \tag{29}$$

Together with (1)–(2), $x(t+1)$ can be expressed as

$$x(t+1) = \Theta_x^*\phi_x(t). \tag{30}$$

Then, the output dynamics (25) can be expressed as

$$y(t+2) = \theta_y^{*T}\Phi_f(\Theta_x^*\phi_x(t)), \tag{31}$$

where $\Phi_f(\Theta_x^*\phi_x(t)) = \phi_f(t+1)$, that is, $\Phi_f(x(t)) = \phi_f(t)$. Note that (31) not only contains linearly parameterized uncertainty θ_y^*, but also contains nonlinearly parameterized uncertainty Θ_x^*.

Auxiliary parameterized signals. To estimate θ_y^* and Θ_x^* in (31), we introduce two auxiliary signals:

$$v(t-1) = \theta_y^{*T}\phi_f(t-1) \in \mathbb{R}, \tag{32}$$
$$w(t-1) = [w_1(t-1), \ldots, w_n(t-1)]^T = \Theta_x^*\phi_x(t-1) \in \mathbb{R}^n. \tag{33}$$

Comparing (31)–(32), we see that $v(t-1) = y(t)$, which implies that $v(t-1)$ is available. Moreover, comparing (30) and (33), we see that $w(t-1) = x(t)$, which implies $w(t-1)$ is also available. With the parametrizations of $v(t-1)$ and $w(t-1)$, we will use gradient algorithms to achieve estimates of θ_y^* and Θ_x^*.

Parameter update laws. Based on the definition of Θ_x^* in (26), we define $\theta_{x_i}^* = [\theta_i^{1*T}, \theta_i^{2*T}]^T$, $i = 1, 2, \ldots, n$. The estimation errors are defined as

$$\epsilon_y(t) = y(t) - \theta_y^T(t)\phi_f(t-1), \tag{34}$$

$$\epsilon_{x_i}(t) = x_i(t) - \theta_{x_i}^T(t)\phi_{x_i}(t-1),\ i = 1, 2, \ldots, n, \tag{35}$$

where $\theta_y(t), \theta_{x_i}(t)$ are the estimates of θ_y^*, $\theta_{x_i}^*$, respectively. Then, the adaptive update laws are designed as

$$\theta_y(t+1) = \theta_y(t) + \frac{\Gamma_y\phi_f(t-1)\epsilon_y(t)}{m_y^2(t-1)} + d_y(t), \tag{36}$$
$$\theta_{x_i}(t+1) = \theta_{x_i}(t) + \frac{\Gamma_{x_i}\phi_{x_i}(t-1)\epsilon_{x_i}(t)}{m_{x_i}^2(t-1)} + d_{x_i}(t), \tag{37}$$

where $\Gamma_k = \mathrm{diag}\{\alpha_{k1}, \ldots, \alpha_{kj}\}$, $k = y, x_i$; $i = 1, 2, \ldots, n$; $j = \sum_i^n p_i, p_i + q_i$, are constant gain matrices such that each diagonal element belongs to $(0, 2)$. Moreover,

$$m_y(t-1) = \sqrt{1 + \phi_f^T(t-1)\phi_f(t-1)}, \tag{38}$$
$$m_{x_i}(t-1) = \sqrt{1 + \phi_{x_i}^T(t-1)\phi_{x_i}(t-1)}; \tag{39}$$

and $d_v(t)$ and $d_{x_i}(t)$ are modification terms which are used to guarantee that $\theta_y(t)$ and $\theta_{x_i}(t)$ stay in some certain regions in the process of parameter adaptation.

The design of $d_f(t)$ and $d_{x_i}(t)$ has the following details. Letting θ_{yj}^* and $\theta_{x_ij}^*$ be the jth components of θ_y^* and $\theta_{x_i}^*$, respectively, we choose some groups of intervals $[\theta_{yj}^a, \theta_{yj}^b]$ and $[\theta_{x_ij}^a, \theta_{x_ij}^b]$ such that $\theta_{yj}^* \in [\theta_{yj}^a, \theta_{yj}^b]$ and $\theta_{x_ij}^* \in [\theta_{x_ij}^a, \theta_{x_ij}^b]$. Then, for

$$d_y(t) = [d_{y1}(t), d_{y2}(t), \ldots, d_{y\sum_{s=1}^n p_s}(t)]^T, \tag{40}$$
$$d_{x_i}(t) = [d_{x_i1}(t), d_{x_i2}(t), \ldots, d_{x_i(p_i+q_i)}(t)]^T, \tag{41}$$

we design

$$d_{kj}(t) = \begin{cases} 0, & \text{if } \theta_{kj}(t) \in [\theta_{kj}^a, \theta_{kj}^b], \\ \theta_{kj}^b - \theta_{kj}(t) - p_{kj}(t), & \text{if } \theta_{kj}(t) + p_{kj}(t) > \theta_{kj}^b, \\ \theta_{kj}^a - \theta_{kj}(t) - p_{kj}(t), & \text{if } \theta_{kj}(t) + p_{kj}(t) < \theta_{kj}^a, \end{cases} \tag{42}$$

where $k = y, x_i$; $j = 1, 2, \ldots, \sum_{s=1}^n p_s$ or $p_i + q_i$; $i = 1, 2, \ldots, n$; $p_{yj}(t)$ and $p_{x_ij}(t)$ are the jth components of φ_y and φ_{x_i} respectively, with $\varphi_y(t) = \frac{\Gamma_y\phi_y(t-1)\epsilon_y(t)}{m_y^2(t-1)}$ and $\varphi_{x_i}(t) = \frac{\Gamma_{x_i}\phi_{x_i}(t-1)\epsilon_{x_i}(t)}{m_{x_i}^2(t-1)}$. The intervals $[\theta_{yj}^a, \theta_{yj}^b]$ and $[\theta_{x_ij}^a, \theta_{x_ij}^b]$ for parameter projection have not been specified yet. Such intervals will be specified later.

The parameter estimates have the following properties:

Lemma 7. *The parameter update laws (36)–(37) ensure (i) $\theta_y(t) \in L^\infty, \theta_{x_i}(t) \in L^\infty, \theta_{kj}(t) \in [\theta_{kj}^a, \theta_{kj}^b]$; and (ii) $\frac{\epsilon_y(t)}{m_y(t-1)} \in L^2 \cap L^\infty$, $\frac{\epsilon_{x_i}(t)}{m_{x_i}(t-1)} \in L^2 \cap L^\infty$, where $k = y, x_i$; $j = 1, 2, \ldots, \sum_{s=1}^n p_s$ or $p_i + q_i$; and $i = 1, 2, \ldots, n$,*

Based on a similar procedure with the proof of Lemma 3.7 in Tao (2003), one can verify that (i) and (ii) hold.

Next, we will use $\theta_y(t)$ and $\Theta_x(t)$ to estimate several future time signals.

Estimation of $y(t+1)$, $\epsilon_y(t+1)$, $x(t+1)$, $d_y(t+1)$ and $\theta_y(t+2)$. Using $\theta_y(t)$, we construct an estimate of $y(t+1)$ as

$$\hat{y}(t+1) = (\theta_y(t) + \frac{\Gamma_y\phi_f(t-1)\epsilon_y(t)}{m_y^2(t-1)} + d_y(t))^T\phi_f(t). \tag{43}$$

Based on (34), we derive an estimate of $\epsilon_y(t+1)$ as

$$\hat{\epsilon}_y(t+1) = -(\theta_y(t) + \frac{\Gamma_y \phi_f(t-1)\epsilon_y(t)}{m_y^2(t-1)} + d_y(t))^T \phi_f(t) + \hat{y}(t+1). \quad (44)$$

Using $\theta_{x_i}(t)$, we construct estimates of $x_i(t+1)$, $i = 1, \ldots, n$, as

$$\hat{x}_i(t+1) = (\theta_{x_i}(t) + \frac{\Gamma_{x_i} \phi_{x_i}(t-1)\epsilon_{x_i}(t)}{m_{x_i}^2(t-1)} + d_{x_i}(t))^T \phi_{x_i}(t). \quad (45)$$

Define

$$\hat{x}(t+1) = [\hat{x}_1(t+1), \ldots, \hat{x}_n(t+1)]^T. \quad (46)$$

Then, $\hat{x}(t+1)$ is available at the current time instant. With (40) and (42), we derive an estimate of $d_y(t+1)$ as

$$\hat{d}_y(t+1) = [\hat{d}_{y1}(t+1), \ldots, \hat{d}_{y\sum_{s=1}^n p_s}(t+1)]^T \quad (47)$$

with

$$\hat{d}_{yj}(t+1) = \begin{cases} 0, & \text{if } \hat{\theta}_{yj}(t+1) \in [\theta_{yj}^a, \theta_{yj}^b], \\ \theta_{yj}^b - \hat{\theta}_{yj}(t+1) - \hat{p}_{yj}(t+1), & \text{if } \hat{\theta}_{yj}(t+1) + \hat{p}_{yj}(t+1) > \theta_{yj}^b, \\ \theta_{yj}^a - \hat{\theta}_{yj}(t+1) - \hat{p}_{yj}(t+1), & \text{if } \hat{\theta}_{yj}(t+1) + \hat{p}_{yj}(t+1) < \theta_{yj}^a, \end{cases}$$

where $\hat{\theta}_{yj}(t+1)$ is the jth component of $\theta_y(t) + \frac{\Gamma_y\phi_f(t-1)\epsilon_y(t)}{m_y^2(t-1)} + d_y(t)$, and $\hat{p}_{yj}(t+1)$ is the jth component of $\hat{\varphi}_y(t+1)$ with $\hat{\varphi}_y(t+1) = \frac{\Gamma_y\phi_y(t)\hat{\epsilon}_y(t+1)}{m_y^2(t)}$. Now, we derive an estimate of $\theta_y(t+2)$ as

$$\hat{\theta}_y(t+2) = \theta_y(t) + \frac{\Gamma_y \phi_f(t-1)\epsilon_y(t)}{m_y^2(t-1)} + d_y(t) + \frac{\Gamma_y \phi_f(t)\hat{\epsilon}_y(t+1)}{m_y^2(t)} + \hat{d}_y(t+1). \quad (48)$$

So far, we have derived the signal estimates $\hat{y}(t+1)$ in (43), $\hat{\epsilon}_y(t+1)$ in (44), $\hat{x}(t+1)$ in (45)–(46), $\hat{d}_y(t+1)$ in (47), and $\hat{\theta}_y(t+2)$ in (48), which are all available.

Adaptive control law. In order to derive an adaptive control law, using available signals and parameter estimates, we plan to construct an implicit function equation with respect to $u(t)$, and show that such an equation has a unique solution which is the desired adaptive control law. The procedure consists of three steps: (i) construct an auxiliary signal using available signals and parameter estimates; (ii) specify $[\theta_{yj}^a, \theta_{yj}^b]$ and $[\theta_{x_ij}^a, \theta_{x_ij}^b]$ in parameter projections of the parameter update laws; and (iii) construct an auxiliary implicit function equation to derive a unique adaptive control law.

Step 1: Construction of an auxiliary signal. With (46) and (48), the auxiliary signal is specified as

$$h(t) = \hat{\theta}_y^T(t+2)\Phi_f(\hat{x}(t+1)) \quad (49)$$

where

$$\Phi_f(\hat{x}(t+1)) = [\bar{f}_1^T(\hat{x}(t+1)), \ldots, \bar{f}_n^T(\hat{x}(t+1))]^T. \quad (50)$$

Note that $h(t)$ is available at the current time instant. With (46) and (48), one can verify that $h(t)$ only depends on $\{\theta_y(t), \theta_{x_i}(t), x(t-1), x(t), y(t-1), y(t), u(t-1), u(t)\}$. For convenience, let

$$\bar{x}_t = [x^T(t-1), x^T(t)]^T, \bar{y}_t = [y(t-1), y(t)]^T,$$
$$\Theta_t = [\theta_y^T(t), \theta_{x_1}^T(t), \ldots, \theta_{x_n}^T(t)]^T, u_{t-1} = u(t-1).$$

Then, $h(t)$ can be expressed as

$$h(t) = H(\bar{x}_t, \bar{y}_t, \Theta_t, u_{t-1}, u(t)). \quad (51)$$

For $h(t)$, we give the following key property.

Lemma 8. *There exist constant intervals $[\tau_{yj}^a, \tau_{yj}^b]$ and $[\tau_{x_ik}^a, \tau_{x_ik}^b]$ such that if $\theta_{yj}(t) \in [\tau_{yj}^a, \tau_{yj}^b]$ and $\theta_{x_ij}(t) \in [\tau_{x_ij}^a, \tau_{x_ij}^b]$, $j = 1, 2, \ldots, \sum_{s=1}^n p_s$ or $p_i + q_i$, $i = 1, 2, \ldots, n$, then*

$$\frac{\partial H(\bar{x}_t, \bar{y}_t, \Theta_t, u_{t-1}, u(t))}{\partial u(t)} \geq \varepsilon_0, \ \forall t \geq 1. \quad (52)$$

where ε_0 is some positive constant.

Under Assumption 2, Lemma 8 is not difficult to obtain, so we omit the proof for simplicity. The intervals $[\tau_{yj}^a, \tau_{yj}^b]$ and $[\tau_{x_ik}^a, \tau_{x_ik}^b]$ are not unique.

Step 2: Specification of $[\theta_{yj}^a, \theta_{yj}^b]$ and $[\theta_{x_ij}^a, \theta_{x_ij}^b]$. Now, based on Lemma 8, we specify $[\theta_{yj}^a, \theta_{yj}^b]$ and $[\theta_{x_ij}^a, \theta_{x_ij}^b]$. Thus, we need the following assumption:

Assumption 4. A group of $[\tau_{yj}^a, \tau_{yj}^b]$, $[\tau_{x_ij}^a, \tau_{x_ij}^b]$ in Lemma 8, $i = 1, \ldots, n$, $j = 1, \ldots, \sum_{s=1}^n p_s$ or $p_i + q_i$ is available.

Remark 9. For adaptive control of canonical-form nonlinear systems, to ensure the adaptive control gain being non-zero, corresponding design conditions should be made. It commonly assumes that some certain intervals are known such that, if the parameter estimates stay in such intervals, the adaptive control gain is always non-zero. Based on this design condition, the parameter projection is added to the parameter update laws which make the parameter estimates always stay in some certain intervals. In this paper, for adaptive control of non-canonical form DT nonlinear systems, Assumption 4 is made, which can be seen as an extension of that for adaptive control of canonical-form nonlinear systems, and is also used for the parameter projection design. Lemma 8 ensures that there exist infinite intervals to ensure that (52) holds. Although the nominal values of θ_y^* and $\theta_{x_i}^*$ may be difficult to obtain, we can acquire some estimates of them using parameter identification algorithms. The derived estimates are used to construct appropriate intervals to ensure that (52) holds. Thus, Assumption 4 is reasonable. The simulation study will illustrate that how to verify Assumption 4 in detail. □

Remark 10. If the system input has enough frequencies so as to make $\phi_f(t)$ and $\phi_{x_i}(t)$ to be persistently exciting, then Assumption 4 is not needed. This is because, under a persistently exciting condition, the parameter estimates can converge to their nominal values. However, the persistently exciting condition is much more restrictive than Assumption 4. □

Under Assumption 4, $[\theta_{yj}^a, \theta_{yj}^b]$ and $[\theta_{x_ij}^a, \theta_{x_ij}^b]$ in $d_j(t)$ are determined as $[\tau_{yj}^a, \tau_{yj}^b]$ and $[\tau_{x_ij}^a, \tau_{x_ij}^b]$, respectively, based on which the adaptive update laws (36)–(37) ensure that (52) always holds.

Step 3: Derivation of an adaptive control law. Using (51), at each time $t \geq 1$, we construct the following equation with respect to $u(t)$ of the form

$$H(\bar{x}_t, \bar{y}_t, \Theta_t, u_{t-1}, u(t)) - y_m(t+2) = 0. \quad (53)$$

To ensure that (53) is a well-defined implicit function equation with respect to $u(t)$, the reference output signal $y_m(t+2)$ needs to satisfy that $y_m(t+2)$ belongs to the range of $H(\bar{x}_t, \bar{y}_t, \Theta_t, u_{t-1}, u(t))$ for each $t \geq 1$.

The following lemma specifies a key property of (53).

Lemma 11. *At each time $t \geq 1$, there exists a unique solution to (53) with respect to $u(t)$.*

Proof. [1] From (52), for all $u \in \mathbb{R}$, under Assumption 4, Lemma 8 implies that there exists a bounded signal denoted as $\underline{\kappa}(t)$ such

[1] To reduce the notation of this paper, this proof uses some notation which have been used in the proof of Theorem 1.

that

$$\underline{\kappa}(t) = \inf_{u\in\mathbb{R}} \left\{ \frac{\partial H(\bar{x}_t, \bar{y}_t, \Theta_t, u_{t-1}, u)}{\partial u} \right\} > 0. \quad (54)$$

Let $c(t) = H(\bar{x}_t, \bar{y}_t, u_{t-1}, \Theta_t, u)|_{u=0} - y_m(t+2)$. If $c(t) = 0$, then $u = 0$ is the solution. Otherwise, consider a compact set defined as

$$\Omega_t = \{u \,|\, |u| \le |c(t)|/\underline{\kappa}(t)\}. \quad (55)$$

Since $\frac{\partial H(\bar{x}_t, \bar{y}_t, \Theta_t, u_{t-1}, u)}{\partial u}$ is continuous on Ω_t with respect to u, we obtain $\frac{\partial H(\bar{x}_t, \bar{y}_t, \Theta_t, u_{t-1}, u)}{\partial u}$ has a maximum value for all $u \in \Omega_t$. Thus, at each time t, for all $u \in \Omega_t$, fixing other signals, there exists a bounded signal denoted as $\bar{\kappa}(t)$ such that

$$\bar{\kappa}(t) = \max_{u\in\Omega_t} \left\{ \frac{\partial H(\bar{x}_t, \bar{y}_t, \Theta_t, u_{t-1}, u)}{\partial u} \right\} > 0.$$

Then, we introduce a mapping: $f_t : \Omega_t \to \mathbb{R}$ defined as

$$f_t(u) = u - \frac{1}{\gamma(t)}(H(\bar{x}_t, \bar{y}_t, \Theta_t, u_{t-1}, u) - y_m(t+2)), \quad (56)$$

where $\gamma(t)$ is a design parameter depending on t such that $\gamma(t) > \bar{\kappa}(t)$. Then, following a procedure similar to the proof of Theorem 5, we have that f_t is a contraction mapping, and there exists a unique solution u to (53). ∇

The solution u to (53) is denoted as u_t^*. One can verify that u_t^* at most depends on $\bar{x}_t$, $\bar{y}_t$, Θ_t, u_{t-1}, $y_m(t+2)$. Thus, u_t^* can be expressed as

$$u_t^* = H_u(\bar{x}_t, \bar{y}_t, \Theta_t, u_{t-1}, y_m(t+2)), \quad (57)$$

where H_u is a nonlinear function. All parameters and signals in (53) are known, in the sense that Eq. (53) is solvable. With c denoting a generic signal bound, we give the following result.

Lemma 12. *The adaptive control law (57) ensures $H(\bar{x}_t, \bar{y}_t, \Theta_t, u_{t-1}, u_t^*) - y_m(t+2) = 0$ for all $t \ge 1$ and satisfies $|u_t^*| \le c\|x(t)\| + c$.*

Proof. In the proof of Lemma 11, we have shown that u_t^* is the unique solution to Eq. (53). Thus, the equation $H(\bar{x}_t, \bar{y}_t, \Theta_t, u_{t-1}, u_t^*) - y_m(t+2) = 0$ always holds for all $t \ge 1$. On the other hand, the mapping f_t defined in (56) always maps u into Ω_t defined in (55). Thus, $u^* \in \Omega_t$. With $c(t) = H(\bar{x}_t, \bar{y}_t, \Theta_t, u_{t-1}, u)|_{u=0} - y_m(t+2)$ and the fact that $\underline{\kappa}(t)$ is away from zero, we have $|u_t^*| \le c|H(\bar{x}_t, \bar{y}_t, \Theta_t, u_{t-1}, 0)| + c$, where c denotes a generic signal bound. From (49) and (51), we have

$$|u_t^*| \le c|\hat{\theta}_y^T(t+2)\Phi_f(\hat{x}(t+1))|_{u(t)=0}| + c. \quad (58)$$

From the structures of $\hat{\theta}_y(t+2)$ and $\hat{x}(t+1)$ in (45) and (48)–(46), respectively, we derive that $\hat{\theta}_y^T(t+2) \in L^\infty$ and $\|\hat{x}(t+1)|_{u(t)=0}\| \le c\|\phi_x(t)|_{u(t)=0}\|$. With $\phi_f(t)$ and $\phi_x(t)$ in (27) and (29), we obtain $\|\phi_x(t)|_{u(t)=0}\| = \|\phi_f(t)\|$. Hence, $\|\hat{x}(t+1)|_{u(t)=0}\| \le c\|\phi_f(t)\|$. Together with (58) and $\hat{\theta}_y^T(t+2) \in L^\infty$, under Assumption 3(a), it is straightforward to obtain $|u_t^*| \le c\|x(t)\| + c$. ∇

Stability and output tracking analysis. Now, we give the main result of this paper.

Theorem 13. *Under Assumptions 1–4, the adaptive control law (57) with the parameter update laws (36)–(37), applied to the system (1) with $\rho = 2$ and unknown C, θ_{ij}^{1*}, θ_{ij}^{2*}, $j = 1, 2, \ldots, p_i(q_i)$, $i = 1, 2, \ldots, n$, ensures closed-loop stability and asymptotic output tracking: $\lim_{t\to\infty}(y(t) - y_m(t)) = 0$.*

Proof. To simplify the notation, we always let $u(t) = u_t^*$ in the followings. With Lemma 12, $\phi_f(t+1) = \Phi_f(x(t+1))$, $H(\bar{x}_t, \bar{y}_t, \Theta_t, u_{t-1}, u_t^*) = \hat{\theta}_y^T(t+2)\Phi_f(\hat{x}(t+1))$, and $y(t+2) = \theta_y^{*T}\phi_f(t+1)$, we have

$$\begin{aligned} &e(t+2) = y(t+2) - y_m(t+2) \\ &= \theta_y^{*T}\phi_f(t+1) - \hat{\theta}_y^T(t+2)\Phi_f(\hat{x}(t+1)) \\ &= \theta_y^T(t+2)\Phi_f(\hat{x}(t+1)) - \hat{\theta}_y^T(t+2)\Phi_f(\hat{x}(t+1)) \\ &\quad + \theta_y^T(t+2)\Phi_f(x(t+1)) - \theta_y^T(t+2)\Phi_f(\hat{x}(t+1)) \\ &\quad + \epsilon_y(t+2). \end{aligned} \quad (59)$$

With (34), (43), (44) and (48), we have

$$\begin{aligned} &\|\theta_y(t+2) - \hat{\theta}_y(t+2)\| \\ &\le c\frac{|\epsilon_y(t+1) - \hat{\epsilon}_y(t+1)|}{m_y(t)} = c\frac{|y(t+1) - \hat{y}(t+1)|}{m_y(t)} \\ &= c\frac{|\theta_y^{*T}\phi_f(t) - \theta_y^T(t+1)\phi_f(t)|}{m_y(t)} = \frac{c|\epsilon_y(t+1)|}{m_y(t)}, \end{aligned}$$

where c denotes a generic signal bound. With the definition of Φ_f in (50), under Assumption 3, Φ_f is globally Lipschitz. Thus, with $\Phi_f(0) = 0$, we have

$$\|\Phi_f(\hat{x}(t+1))\| \le c\|\hat{x}(t+1)\| \le c\|\phi_x(t)\|. \quad (60)$$

With (35) and (45)–(46), we derive $\|\Phi_f(x(t+1)) - \Phi_f(\hat{x}(t+1))\| \le c\|x(t+1) - \hat{x}(t+1)\| \le c\|\epsilon_x(t+1)\|$, where $\epsilon_x(t+1) = [\epsilon_{x_1}(t+1), \ldots, \epsilon_{x_n}(t+1)]^T$. From (28), (35), together with the system model (1), we get

$$\|\phi_x(t)\| \le c\|\phi_f(t)\| + c\|\phi_g(t)\|\|u_t^*\|. \quad (61)$$

Under Assumption 3, we see that $\|\phi_f(t)\| \le c\|x(t)\|$ and $\|\phi_g(t)\| \in L^\infty$. Thus, based on Lemma 12, (61) implies $\|\phi_x(t)\| \le c\|x(t)\| + c$. With (39) and (60), we obtain

$$m_{x_i}(t) \le c\|x(t)\| + c, \ \|\Phi_f(\hat{x}(t+1))\| \le c\|x(t)\| + c.$$

From (38), we get

$$\begin{aligned} m_f(t+1) &\le c\|\phi_f(x(t+1))\| + c \le c\|x(t+1)\| + c \\ &\le c\|\phi_x(t)\| + c \le c\|x(t)\| + c. \end{aligned}$$

From the diffeomorphism $T(x) = [\xi^T, \eta^T]^T$, we have $\|x(t)\| \le c\|\xi(t)\| + c\|\eta(t)\|$ with $\xi(t) = [y(t), y(t+1)]^T$. Under Assumption 1, we derive that $\|\eta(t)\| \le c\|\xi(t)\| + c$. Therefore, with $e(t) = y(t) - y_m(t)$, $\xi(t) = [y(t), y(t+1)]^T$ and $y_m(t) \in L^\infty$, we derive that

$$\|x(t)\| \le c\|\xi(t)\| + c \le c\max_{i=0,1}|e(t+i)| + c.$$

With the above derivations, (59) implies that

$$\begin{aligned} |e(t+2)| \le\ & \frac{c|\epsilon_y(t+1)|}{m_y(t)}\|\phi_x(t)\| + c\frac{\|\epsilon_x(t+1)\|}{\|m_x(t)\|}\|m_x(t)\| \\ & + \frac{|\epsilon_y(t+2)|}{m_f(t+1)}m_f(t+1), \end{aligned} \quad (62)$$

where $m_x(t) = [m_{x_1}(t), \ldots, m_{x_n}(t)]^T$. From Lemma 7, we obtain that the signals $\frac{|\epsilon_y(t+1)|}{m_y(t)}$, $\frac{\|\epsilon_x(t+1)\|}{\|m_x(t)\|}$, $\frac{|\epsilon_y(t+2)|}{m_f(t+1)}$ all decay to zero asymptotically.

Hence, letting $\mu(t)$ denote a generic asymptotically decaying signal, we have $|e(t+2)| \le \mu(t)\max_{i=0,1}|e(t+i)| + \mu(t) + c$ which implies that $e(t) \in L^\infty$. Thus, $y(t) \in L^\infty$ and $y(t+1) \in L^\infty$, which further implies $\xi(t) = [y(t), y(t+1)]^T \in L^\infty$. Under Assumption 1, we obtain $\eta(t) \in L^\infty$.

Then, based on the diffeomorphism $T(x) = [\xi^T, \eta^T]^T$, it follows that $x(t) \in L^\infty$. Based on Lemma 12, we derive $u(t) \in L^\infty$. So far, all closed-loop signals are bounded. Thus, $\phi_x(t)$, $m_x(t)$ and $m_f(t+1)$ are all bounded. From (62), we conclude that $\lim_{t\to\infty} e(t) = 0$. ∇

Theorem 13 provides a fundamental adaptive control method for the system (1) with relative degree two which successfully solves the technical problems of nonlinear parametrization and non-affine control input.

5.3. Iterative solution based adaptive control design

In some cases, Eq. (53) may be too complicated to get an analytical solution. Alternatively, we will develop an iterative solution based adaptive control law.

Iterative controller structure. The iterative adaptive law is designed as

$$u_i(t) = u_{i-1}(t) - \frac{H(\bar{x}_t, \bar{y}_t, \Theta_t, u_{t-1}, u_{i-1}(t)) - y_m(t+2)}{\gamma(t)}, \quad (63)$$

where $u_0(t) = u_{t-1}$ is the control signal at the time instant $t-1$ and $\gamma(t)$ has been clarified below (56).

For the control law (63), we give the following result.

Theorem 14. *Under Assumptions 1–4, for all $t \geq 1$, $\{u_i(t)\}$ in (63) is convergent, i.e., $\lim_{i\to\infty} u_i(t) = u_t^*$, where u_t^* is the unique solution to (53).*

Proof. Based on (54), (56), and (63), we derive that

$$|u_{m+1}(t) - u_{n+1}(t)| \leq \left(1 - \frac{\underline{\kappa}(t)}{\gamma(t)}\right)|u_m(t) - u_n(t)|, \quad (64)$$

$\forall m, n \in \{0, 1, 2, \ldots\}$. Using Cauchy Convergence Criterion, it follows from (64) that $\{u_i(t)\}$ is convergent. Then, letting $i \to \infty$ of the two sides of (63), we have $u_t^* = u_t^* - H(\bar{x}_t, \bar{y}_t, \Theta_t, u_{t-1}, u_t^*) - y_m(t+2)/\gamma(t)$, that is, $H(\bar{x}_t, \bar{y}_t, \Theta_t, u_{t-1}, u_t^*) - y_m(t+2) = 0$. Together with Lemma 11, u_t^* is the unique solution to (53). ∇

Since u_t^* is a limitation, it is often not achievable. To implement the iterative control method, we choose $u_{p(t)}(t)$ generated by $p(t)$ iterations of $\{u_i(t)\}$ defined in (63) as the adaptive law to guarantee

$$|y(t+2) - y_m(t+2)| \leq \epsilon + \delta, \quad \forall t \geq 1, \quad (65)$$

where ϵ is a given admissible error and δ is an asymptotically decaying signal. Now, we give the following result.

Theorem 15. *Under Assumptions 1–4, the adaptive law $u_{p(t)}(t)$ with the update laws (36)–(37), applied to the system (1) with $\rho = 2$ and unknown C, $\theta_{ij}^{1*}, \theta_{ij}^{2*}$, $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, p_i(q_i)$, ensures closed-loop stability and (65) if $p(t) = 0$ for the case of $u_1(t) = u_0(t)$ or*

$$p(t) \geq \log_{\frac{\gamma(t)-\underline{\kappa}(t)}{\gamma(t)}} \frac{\underline{\kappa}(t)\epsilon}{2\gamma(t)\bar{\kappa}(t)|u_1(t) - u_0(t)|}$$

for the case of $u_1(t) \neq u_0(t)$.

Proof. If $u_1(t) = u_0(t)$, from (63), we see that

$$H(\bar{x}_t, \bar{y}_t, \Theta_t, u_{t-1}, u_0(t)) - y_m(t+2) = 0$$

which implies $u_0(t)$ is the desired control signal at the time t. Otherwise, from (64), by induction, it yields $|u_{p(t)+1}(t) - u_{p(t)}(t)| \leq (1 - \frac{\underline{\kappa}(t)}{\gamma(t)})^{p(t)}|u_1(t) - u_0(t)|$. Then,

$$\begin{aligned} &|u_{p(t)+q(t)}(t) - u_{p(t)}(t)| \\ &\leq \frac{\gamma(t)}{\underline{\kappa}(t)}\left(1 - \frac{\underline{\kappa}(t)}{\gamma(t)}\right)^{p(t)}\left(1 - \left(1 - \frac{\underline{\kappa}(t)}{\gamma(t)}\right)^{q(t)}\right)|u_1(t) - u_0(t)|. \end{aligned}$$

Letting $q(t) \to \infty$ yields

$$|u_t^* - u_{p(t)}(t)| = \frac{\gamma(t)}{\underline{\kappa}(t)}\left(1 - \frac{\underline{\kappa}(t)}{\gamma(t)}\right)^{p(t)}|u_1(t) - u_0(t)|.$$

With $H(\bar{x}_t, \bar{y}_t, \Theta_t, u_{t-1}, u_t^*) - y_m(t+2) = 0$, applying the Mean Value Theorem to it yields

$$\begin{aligned} &|H(\bar{x}_t, \bar{y}_t, \Theta_t, u_{t-1}, u_{p(t)}(t)) - y_m(t+2)| \\ &\leq \left|\frac{\partial H(\bar{x}_t, \bar{y}_t, \Theta_t, u_{t-1}, u_{p(t)}(t))}{\partial u_{p(t)}(t)}|_{u_{p(t)}(t)=\psi_0}\right| \cdot |u_{p(t)}(t) - u_t^*| \\ &\leq \bar{\kappa}(t)|u_{p(t)}(t) - u_t^*|, \end{aligned}$$

where ψ_0 is some value between $u_{p(t)}(t)$ and u_t^*. Then,

$$\begin{aligned} &|H(\bar{x}_t, \bar{y}_t, \Theta_t, u_{t-1}, u_{p(t)}(t)) - y_m(t+2)| \\ &\leq \left(1 - \frac{\underline{\kappa}(t)}{\gamma(t)}\right)^{p(t)} \frac{\gamma(t)\bar{\kappa}(t)|u_1(t) - u_0(t)|}{\underline{\kappa}(t)}. \end{aligned}$$

Now, let $\left(1 - \frac{\underline{\kappa}(t)}{\gamma(t)}\right)^{p(t)} \frac{\gamma(t)\bar{\kappa}(t)|u_1(t)-u_0(t)|}{\underline{\kappa}(t)} \leq \frac{1}{2}\epsilon$, then

$$p(t) \geq \log_{\frac{\gamma(t)-\underline{\kappa}(t)}{\gamma(t)}} \frac{\underline{\kappa}(t)\epsilon}{2\gamma(t)\bar{\kappa}(t)|u_1(t) - u_0(t)|}. \quad (66)$$

Thus, as long as (66) holds, $|H(\bar{x}_t, \bar{y}_t, \Theta_t, u_{t-1}, u_{p(t)}(t)) - y_m(t+2)| \leq \frac{1}{2}\epsilon$. Moreover,

$$\begin{aligned} &|y(t+2) - y_m(t+2)| \\ &\leq |y(t+2) - H(\bar{x}_t, \bar{y}_t, \Theta_t, u_{t-1}, u_t^*)| \\ &\quad + 2|H(\bar{x}_t, \bar{y}_t, \Theta_t, u_{t-1}, u_{p(t)}(t)) - y_m(t+2)|. \end{aligned}$$

Since $\lim_{t\to\infty}(y(t+2) - H(\bar{x}_t, \bar{y}_t, \Theta_t, u_{t-1}, u_t^*)) = 0$, we see that (65) holds. Following a similar procedure with that in the proof of Theorem 13, we derive that all closed-loop signals are bounded. ∇

Theorem 15 provides a constructive method to implement the iterative solution based adaptive control law.

5.4. Extension to higher-order relative degree case

Now, we show that the proposed method for the $\rho = 2$ case can be extended to any higher-order relative degree case. Since it is quite complicated to analytically describe the general case, we demonstrate the $\rho = 3$ case to verify the above point of view.

For the system (1) with $\rho = 3$, $y(t+2)$ no longer depends on $u(t)$. Thus, from (31), $y(t+2)$ can be expressed as

$$y(t+2) = \theta_y^{*T}\Phi_f(\Theta_x^*\phi_{x0}(t)),$$

where $\phi_{x0}(t) = [\phi_{x_10}^T(t), \ldots, \phi_{x_n0}^T(t)]^T \in \mathbb{R}^{\sum_{i=1}^n(p_i+q_i)}$ with $\phi_{x_i0}(t) = [\bar{f}_i^T(x(t)), \bar{g}_i^T(x(t))\cdot 0]^T \in \mathbb{R}^{p_i+q_i}$. Then, $y(t+3)$ can be expressed as

$$y(t+3) = \theta_y^{*T}\Phi_f(\Theta_x^*\phi_{x0}(t+1)) = \theta_y^{*T}\Phi_f(\Theta_x^*\phi_{x0}(\Theta_x^*\phi_x(t)))$$

such that $\theta_y^{*T}\frac{\partial\Phi_f(\Theta_x^*\phi_{x0}(t+1))}{\partial\phi_{x0}(t+1)}\frac{\partial\phi_{x0}(t+1)}{\partial x(t+1)}g(x(t)) \geq \varepsilon > 0$.

Similar to the relative degree two case, the following implicit function equation should be constructed:

$$\hat{\theta}_y^T(t+3)\Phi_f(\hat{\Theta}_x(t+2)\phi_{x0}(\hat{\Theta}_x(t+1)\phi_x(t))) - y_m(t+3) = 0. \quad (67)$$

To acquire (67), with $\hat{y}(t+1)$, $\hat{\epsilon}_y(t+1)$, $\hat{x}(t+1)$, $\hat{d}_y(t+1)$, $\hat{\theta}_y(t+2)$ specified for the relative degree two case, we need to further specify $\hat{\theta}_y(t+3)$, $\hat{\Theta}_x(t+2)$. From (48), $\hat{\theta}_y(t+3)$ is derived as

$$\begin{aligned} \hat{\theta}_y(t+3) = \theta_y(t) &+ \frac{\Gamma_y\phi_f(t-1)\epsilon_y(t)}{m_y^2(t-1)} + d_y(t) \\ &+ \frac{\Gamma_y\phi_f(t)\hat{\epsilon}_y(t+1)}{m_y^2(t)} + \hat{d}_y(t+1) \end{aligned}$$

$$+\frac{\Gamma_y\hat{\phi}_f(t+1)\hat{\epsilon}_y(t+2)}{\hat{m}_y^2(t+1)}+\hat{d}_y(t+2), \tag{68}$$

where $\hat{\phi}_f(t+1)=[\bar{f}_1^T(\hat{x}(t+1)),\ldots,\bar{f}_n^T(\hat{x}(t+1))]^T$ and $\hat{m}_y(t+1)=\sqrt{1+\hat{\phi}_f^T(t+1)\hat{\phi}_f(t+1)}$. In the right side of (68), only $\hat{\epsilon}_y(t+2)$ and $\hat{d}_y(t+2)$ are yet to be specified. Moreover, as long as $\hat{\epsilon}_y(t+2)$ is specified, $\hat{d}_y(t+2)$ is easy to be derived. To obtain $\hat{\epsilon}_y(t+2)$, from (44), we only need to specify $\hat{y}(t+2)$. From (43), $\hat{y}(t+2)$ is specified as

$$\hat{y}(t+2)=(\theta_y(t+1)+\frac{\Gamma_y\phi_f(t)\hat{\epsilon}_y(t+1)}{m_y^2(t)}+\hat{d}_y(t+1))^T\hat{\phi}_f(t+1)$$

with $\theta_y(t+1)=\theta_y(t)+\frac{\Gamma_y\phi_f(t-1)\epsilon_y(t)}{m_y^2(t-1)}+d_y(t)$. Following a similar way, one can also specify $\hat{\Theta}_x(t+2)$. Then, the implicit function equation (67) can be specifically constructed. Finally, a unique adaptive control law can be derived from (67), which can ensure closed-loop stability and asymptotic output tracking for the system (1) with $\rho=3$. Here, we do not provide the details.

So far, several adaptive control methods for the system (1) with relative degree one and two have been developed. In particular, some extensions of the proposed adaptive control methods are given.

6. Simulation study

An illustrative example is given to show the control design procedure and verify the effectiveness.

6.1. Simulation model

We consider the following system model

$$\begin{aligned}
x_1(t+1)&=1.3x_1(t)+1.2\sin x_1(t)\cos x_3(t)\\
&\quad+(2+2\sin^2(x_1(t)))u(t),\\
x_2(t+1)&=1.2x_2(t)+1.5\arctan x_1(t)+1.3\sin x_3(t),\\
x_3(t+1)&=0.5x_3(t)+1.6\sin x_2(t),
\end{aligned}\tag{69}$$

where $x_i\in\mathbb{R}$, $i=1,2,3$, are three state variables, $u\in\mathbb{R}$ is the system input, and the system output is $y(t)=x_2(t)$. This model is parameterized as

$$x(t+1)=\Theta_f^*\phi_f(t)+\Theta_g^*\phi_g(t)u(t),\ \ y(t)=Cx(t), \tag{70}$$

where $\Theta_g^*=\text{diag}\{\theta_{g1}^*,0,0\}\in\mathbb{R}^{3\times3}$ with $\theta_{g1}^*=2$, $\Theta_f^*=\text{diag}\{\theta_{f1}^{*T},\theta_{f2}^{*T},\theta_{f3}^{*T}\}\in\mathbb{R}^{3\times7}$,

$$\begin{aligned}
\phi_f(t)&=[x_1(t),\sin x_1(t)\cos x_3(t),x_2(t),\arctan x_1(t),\\
&\qquad\sin x_3(t),x_3(t),\sin x_2(t)]^T\in\mathbb{R}^7,\\
\phi_g(t)&=[1+\sin^2(x_1(t)),0,0]^T\in\mathbb{R}^3,
\end{aligned}\tag{71}$$

$\theta_{f1}^*=[1.3,1.2]^T$, $\theta_{f2}^*=[1.2,1.5,1.3]^T$, $\theta_{f3}^*=[0.5,1.6]^T$, and $C=[0,1,0]$. In this simulation, we assume that Θ_f^*, Θ_g^*, C are unknown, and $\phi_f(t)$, $\phi_g(t)$ are known.

6.2. Verification of the assumptions

From (69) and (70), we derive

$$\begin{aligned}
y(t+2)&=1.44x_2(t)+1.8\arctan x_1(t)+1.56\sin x_3(t)\\
&\quad+1.5\arctan\big(1.3x_1(t)+1.2\sin x_1(t)\cos x_3(t)\\
&\quad+(2+2\sin^2(x_1(t)))u(t)\big)\\
&\quad+1.3\sin(0.5x_3(t)+1.6\sin x_2(t))
\end{aligned}$$

which implies that $y(t+2)$ contains both linearly and nonlinearly parameterized uncertainties, and nonlinearly depends on $u(t)$. Let θ_{f21}^* denote the first element of θ_{f2}^*, then the relative degree condition can be verified from

$$\frac{\partial y(t+2)}{\partial x(t+1)}\cdot\frac{\partial x(t+1)}{\partial u(t)}=\frac{\theta_{f21}^*\theta_{g1}^*(1+\sin^2(x_1(t)))}{1+\sin^2(x_1(t+1))} \tag{72}$$

which is always non-zero for all $(x,u)\in\mathbb{R}^3\times\mathbb{R}$.

Thus, the model (69) has relative degree 2, based on which we choose the state transformation as $T(x)=[x_2,1.2x_2+1.5\arctan x_1+1.3\sin x_3,x_3]^T$. One can readily verify that $T(x)$ is smooth with respect to x and the Jacobian matrix $\frac{\partial T(x)}{\partial x}$ is nonsingular for all $x\in\mathbb{R}^3$. Therefore, $T(x)$ is a diffeomorphism, and the internal dynamics is $x_3(t+1)=0.5x_3(t)+1.6\sin x_2(t)$. Since $x_3(t+1)=0.5x_3(t)$ decays to zero exponentially and $1.6\sin x_2$ is Lipschitz with respect to x_2, the internal dynamics is ISS with respect to x_2 as the input. Thus, Assumption 1 is satisfied. It follows from (72) that Assumption 2 is also satisfied. From the structure of the model (69), Assumption 3 is satisfied.

We deduce from (72) that, in the process of parameter adaptation, the relative degree two condition can be verified by judging whether $\frac{\theta_{f21}(t)\theta_{g1}(t)(1+x_1^2(t))}{1+x_1^2(t+1)}$ is zero or not, where $\theta_{f21}(t)$ and $\theta_{g1}(t)$ are the estimates of θ_{f21}^* and θ_{g1}^*, respectively. As long as $\theta_{f21}(t)$ and $\theta_{g1}(t)$ are non-zero, the relative degree two condition always holds. Thus, only two intervals associated with θ_{f21}^* and θ_{g1}^* need to be chosen for the parameter projection design, in the sense that Assumption 4 is easy to be satisfied.

6.3. Construction of an auxiliary function

Before deriving the adaptive control law, we first need to construct Eq. (53), for which we only need to specify $\hat{\theta}_y^T(t+2)\Phi_f(\hat{x}(t+1))$. Now, with $\phi_f(t)$ and $\phi_g(t)$ in (71), we further define

$$\begin{aligned}
&v(t-1)=\theta_y^{*T}\phi_f(t-1),\ w(t-1)=\Theta_x^*\phi_x(t-1),\\
&\theta_y^*=[0,0,1.2,1.5,1.3,0,0]^T,\ \ \theta_{x_1}^*=[1.3,1.2,2]^T,\\
&\theta_{x_2}^*=[1.2,1.5,1.3,0]^T,\ \ \theta_{x_3}^*=[0.5,1.6,0]^T,\\
&\phi_{x_1}(t)=[x_1(t),\sin x_1(t)\cos x_3(t),x_2(t),(1+\sin^2(x_1(t)))u(t)]^T,\\
&\phi_{x_2}(t)=[x_2(t),\arctan x_1(t),\sin x_3(t),0]^T,\\
&\phi_{x_3}(t)=[x_3(t),\sin x_2(t),0]^T.
\end{aligned}$$

Let $\theta_y(t)$ and $\theta_{x_i}(t)$ be the estimates of θ_y^* and $\theta_{x_i}^*$, respectively. Then, $\epsilon_y,\epsilon_{x_i}$ can be specified based on (34) and (35). After that, the parameter update laws for $\theta_y(t)$ and $\theta_{x_i}(t)$ can be specified based on (36)–(37). In particular, under Assumption 4, we can choose the intervals $[\tau_{yj}^a,\tau_{yj}^b]$ and $[\tau_{x_ij}^a,\tau_{x_ij}^b]$ for the parameter projection design. Based on the clarification in the paragraph above Section 6.3, we only need to specify $[\tau_{y3}^a,\tau_{y3}^b]$ and $[\tau_{x_14}^a,\tau_{x_14}^b]$, and other intervals can be chosen arbitrarily. Here, $[\tau_{y3}^a,\tau_{y3}^b]$ and $[\tau_{x_13}^a,\tau_{x_13}^b]$ are chosen as $[0.5,2]$ and $[1,3]$, respectively. So far, the parameter update laws can be completely specified, based on which $\hat{\theta}_y^T(t+2)\Phi_f(\hat{x}(t+1))$ can also be specified. Finally, the implicit function equation (53) can be specified.

6.4. Simulation results

Based on the structure of system nonlinearities, it is not easy to get an analytical solution $u(t)$ from $\hat{\theta}_y^T(t+2)\Phi_f(\hat{x}(t+1))=y_m(t+2)$. In this case, by Theorem 15, we use the iteration solution $u_{p(t)}(t)$ as the adaptive law. Note that such an control law ensures bounded output tracking. The admissible error ϵ in (65) is chosen as 0.001. Two reference output signals are given to demonstrate the closed-loop stability and tracking performance:

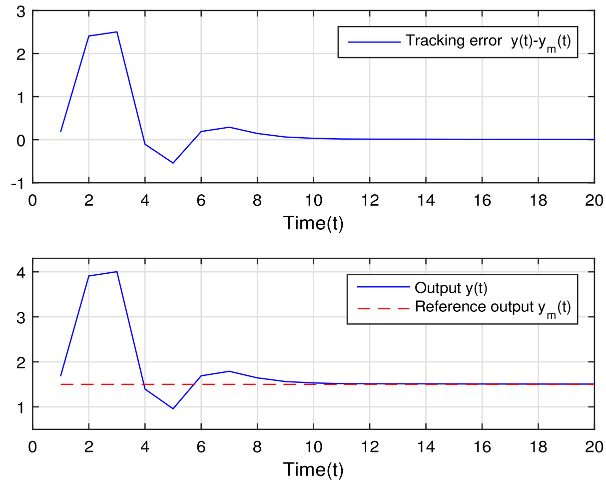

Fig. 1. System output y vs. reference output y_m (Case I).

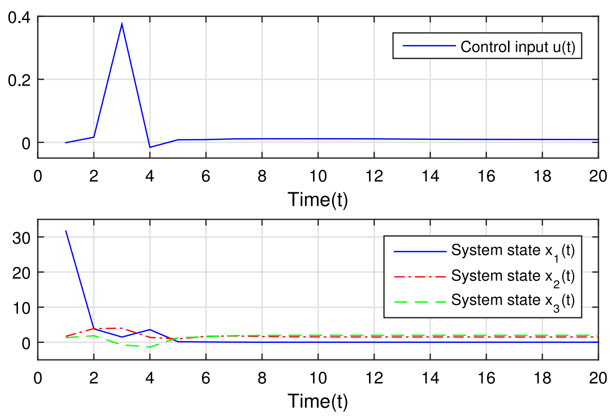

Fig. 2. Control input u and system state x (Case I).

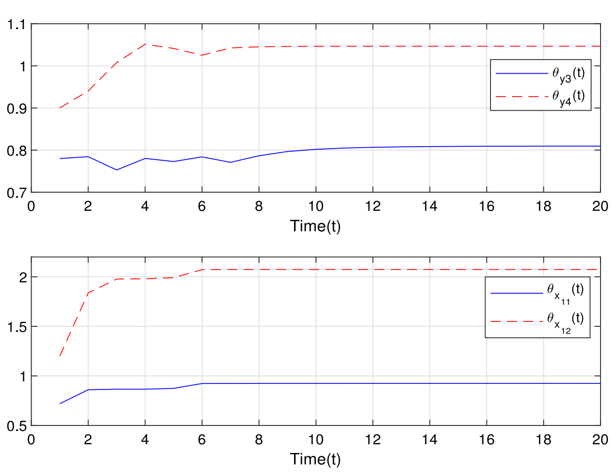

Fig. 3. Adaptation of parameters of θ_y and θ_{x_1} (Case I).

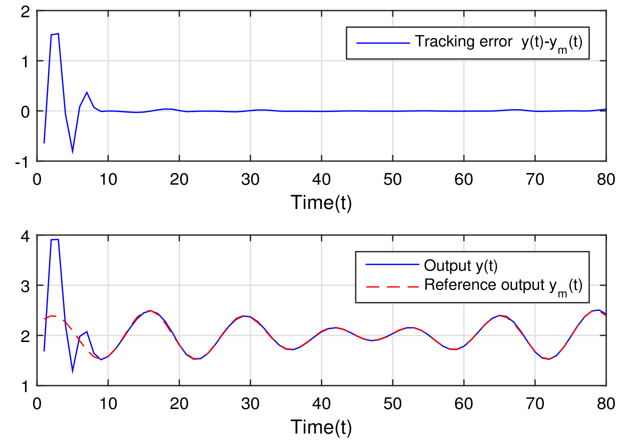

Fig. 4. System output y vs. reference output y_m (Case II).

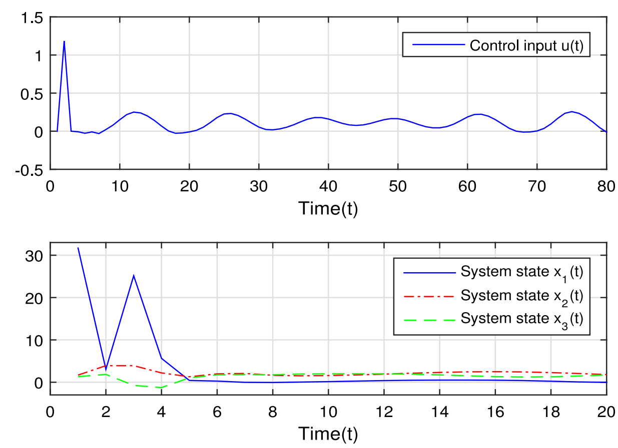

Fig. 5. Control input u and system state x (Case II).

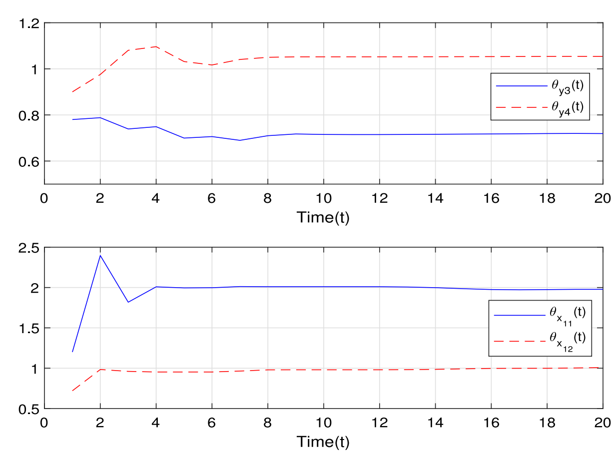

Fig. 6. Adaptation of parameters of θ_y and θ_{x_1} (Case II).

one is a constant signal $y_m = 1.5$; and the other is a time-varying signal $y_m(t) = 2 + 0.3\sin(0.5t) + 0.2\cos(0.4t)$.

For Case I, Fig. 1 shows the output response versus the constant reference output. Fig. 2 demonstrates the input and state response. Fig. 3 presents the parameter adaptation response (due to space limit, only four parameter estimates are given). For Case II, Fig. 4 shows the output response versus the time-varying reference output. Fig. 5 demonstrates the input and state response. Fig. 6 presents the parameter adaptation response (due to space limit, we also give four parameter estimates).

Based on the simulation results, one can see that the system output tracks the reference output signal within the prescribed error ϵ (if an analytical adaptive law u_t^* is specified from the equation $\hat{\theta}_y^T(t+2)\Phi_f(\hat{x}(t+1)) = y_m(t+2)$, asymptotic tracking can be obtained), and the closed-loop signals are all bounded. As illustrated in Figs. 3 and 6, the parameter estimates may not converge to their nominal values (the proposed control method does not depend on the persistently exciting condition, and thus, the parameter estimates generally do not converge to their nominal values). However, the desired system performance is achieved.

7. Concluding remarks

This paper established an adaptive state feedback output tracking control framework for non-canonical form DT nonlinear systems with parametric uncertainties, where several adaptive control methods have been developed based on an implicit function based formulation. Specifically, an adaptive parametric reconstruction based method was proposed to effectively deal with all unknown parameters in the output dynamics, and an implicit function equation was constructed to solve the non-affine control input issue, based on which a unique adaptive control law was derived for the controlled plant to ensure desired system performance. In addition, an iterative solution based adaptive control law was also developed, which provides an alternative method when the analytical adaptive law is difficult to obtain. This paper is the first to systematically address the adaptive control problem of non-canonical form DT nonlinear systems with uncertainties. Future work is needed to address the applications of the proposed adaptive control methods.

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